ISRAEL JOURNAL OF MATHEMATICS **163** (2008), 1–27
DOI: 10.1007/s11856-008-0001-x

EMBEDDINGS OF NON-COMMUTATIVE L^p-SPACES INTO PREDUALS OF FINITE VON NEUMANN ALGEBRAS*

BY

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Dedicated to Professor H. P. Rosenthal on the occasion of his sixty-fifth birthday

ABSTRACT

Let $\mathcal{R}$ be a (not necessarily semi-finite) σ-finite von Neumann algebra. We prove that there exists a finite von Neumann algebra $\mathcal{N}$ so that for every $1 < p < 2$, the Haagerup L^p-space associated with $\mathcal{R}$ embeds isomorphically into $\mathcal{N}_*$. We also provide a proof of the following non-commutative generalization of a classical result of Rosenthal: if $\mathcal{M}$ is a semi-finite von Neumann algebra then every reflexive subspace of $\mathcal{M}_*$ embeds isomorphically into $L^r(\mathcal{M})$ for some $r > 1$.

1. Introduction

In recent years, Banach space structure of non-commutative L^p-spaces and their subspaces have been studied extensively. Recent work in this line of research can be found for instance in [13, 18, 32, 35, 36]. The recent survey article [29] provides an up-to-date information on the latest developments.

The present paper deals with the following general Banach embedding problem: Let M and N be von Neumann algebras. When should one expect that the corresponding L^p-spaces $L^p(M)$ and $L^p(N)$ are linearly isomorphic or that $L^p(M)$ is linearly isomorphic to a subspace of $L^p(N)$? It was shown in [35] that

* Research partially supported by NSF grant DMS-0456781.
Received January 23, 2006

if M is infinite and N is finite then $L^p(M)$ and $L^p(N)$ are not isomorphic for all $0 < p < \infty$, $p \neq 2$. One of the many problems that remain open in this direction is whether the predual of a given type-III von Neumann algebra can be embedded (as Banach space) into the predual of a semi-finite von Neumann algebra. We recall that a recent result of Oikhberg, Rosenthal, and Størmer [27] shows that the answer to the isometric version of the above mentioned problem is negative in the following sense: if M and N are von Neumann algebras with the properties that M is semi-finite and N_* embeds ismetrically into M_* then N is semi-finite.

If one however considers different indices in the L^p-spaces involved then the situation is quite different. This is witnessed by the following rather deep isometric result, due to Junge (see [18, Theorem 0.2, Theorem 0.3] and [17]):

Theorem 1.1 ([18], [17]): *Let N be a semi-finite von Neumann algebra with normal, semi-finite, faithful trace τ. There exists a finite von Neumann algebra M with a normal, tracial, faithful state σ such that for all $0 < q < p < 2$, there exists an isometric embedding of $L^p(N, \tau)$ into $L^q(M, \sigma)$.*

The preceding theorem can be viewed as a non-commutative extension of a very well-studied classical theme of embedding a (commutative) L^p-space into another. We refer to [4] and [24, pp. 181–215] for more information on this line of research. As in the classical case, the method of the proof in [18] is probabilistic. Theorem 1.1 says a lot more than the classical situation. In fact, it allows (isometric) embeddings of L^p-spaces associated with type-II_∞ von Neumann algebras into L^q-spaces associated with type-II_1 von Neumann algebras.

It is a natural question to consider whether or not Theorem 1.1 can be extended to include general von Neumann algebras which are not necessarily semi-finite. We will consider L^p-spaces associated with type-III von Neumann algebras as those introduced by Haagerup in [11] (see the description in the next section). We remark that using the general reduction to finite von Neumann algebra due to Haagerup [12] one can easily deduce from Theorem 1.1 the following.

Corollary 1.2: *Let N be a type-III von Neumann algebra. There exists another von Neumann algebra W such that for all $0 < q < p < 2$, there exists an isometric embedding of $L^p(N)$ into $L^q(W)$.*

Our main goal is to study whether or not the von Neumann W in the corollary above can still be chosen to be finite (or semi-finite). This question is primarily motivated by the fact that as Banach spaces, non-commutative L^p-spaces associated with semi-finite von Neumann algebras are much closer to classical L^p-spaces than those associated with type-III von Neumann algebras. Indeed, several techniques from classical theory of Banach lattices and rearrangement invariant spaces have non-commutative generalizations when working with semi-finite von Neumann algebras. Our main result shows that for the case of isomorphism, the answer is positive. More precisely, we obtain the following result:

THEOREM 1.3: *Let N be a type-III von Neumann. There exists a finite von Neumann algebra M such that for all $0 < q < p < 2$, there exists an isomorphic embedding of $L^p(N)$ into $L^q(M)$.*

We emphasize that the preceding theorem deals only with isomorphic embeddings in contrast to the isometric results of Junge stated in Theorem 1.1 and Corollary 1.2 (see Theorem 3.1 below for details). We also note that for $1 \leq q < p < 2$, Theorem 1.3 does not extend to the case of complete isomorphism.

Our proof relies on the construction of Haagerup L^p-spaces as collection of measurable operators associated with semi-finite von Neumann algebras, the notion of strong embedding related to measure topology on non-commutative spaces, consideration of Rademacher types, and a non-commutative generalization of a classical result of Rosenthal on reflexive subspaces of L^1-spaces. As application of this type of embeddings, we obtain (using a recent result of Junge and Parcet on reflexive subspaces of preduals of von Neumann algebras) that for any given von Neumann algebra, all reflexive subspaces of its predual embed isomorphically into the predual of a finite von Neumann algebra (see Theorem 3.4 below).

The paper is organized as follows. In §2, we set some basic definitions and background on symmetric spaces of measurable operators and Haagerup L^p-spaces along with some preliminary results. In §3, we provide the statement of the main result and its proof. We include as an appendix a detailed treatment of the non-commutative generalization of a classical result of Rosenthal on reflexive subspaces of non-commutative L^1-spaces associated with semi-finite von Neumann algebras.

We use standard notation in Banach spaces and operator algebras. We refer to [20, 21, 37] for background on von Neumann algebra theory and [24, 39] for unexplained notation or terminology from Banach space theory.

2. Definitions and preliminary results

In this section, we collect some basic facts and notation that will be used throughout the paper.

2.1. Some Banach space and Banach lattice concepts. We denote by $(r_n)_{n\geq 1}$ the usual Rademacher sequence in $[0,1]$ defined by setting

$$r_n(t) = \operatorname{sgn}(\sin 2^n \pi t), \quad t \in [0,1], \quad n = 1, 2, \ldots.$$

We recall [24] that a Banach space X is said to have **Rademacher type** p (respectively, **Rademacher cotype** q) for some $1 < p \leq 2$ (respectively, $2 \leq q < \infty$) if there exists a finite constant C such that for every finite sequence $(x_i)_{i=1}^n$ in X,

$$\Big(\int_0^1 \Big\| \sum_{i=1}^n r_i(t) x_i \Big\|_X^2 \, dt\Big)^{1/2} \leq C \Big(\sum_{i=1}^n \|x_i\|_X^p\Big)^{1/p},$$

respectively,

$$\Big(\sum_{i=1}^n \|x_i\|_X^q\Big)^{1/q} \leq C \Big(\int_0^1 \Big\| \sum_{i=1}^n r_i(t) x_i \Big\|_X^2 \, dt\Big)^{1/2}.$$

Usually these notions are referred to in the literature as type p and cotype q respectively but to avoid any confusion with the notion of types for von Neumann algebras, we will keep the term “Rademacher” throughout the paper. The smallest such constant C is called the **Rademacher type** p **constant** (respectively, **Rademacher cotype** q **constant**) of X.

For $0 < \alpha < \infty$, a quasi-Banach lattice E is said to be α**-convex** if there exists a constant C such that for every finite sequence $(x_i)_{i=1}^n$ in E,

$$\Big\| \Big(\sum_{i=1}^n |x_i|^\alpha\Big)^{1/\alpha} \Big\|_E \leq C \Big(\sum_{i=1}^n \|x_i\|_E^\alpha\Big)^{1/\alpha},$$

The least of such constant C is called the α**-convexity constant** of E. The quasi-Banach lattice E is said to satisfy **a lower** q**-estimate** if there exists a positive constant $K > 0$ such that for all finite sequence $(x_i)_{i=1}^n$ of mutually disjoint elements of E,

$$\Big(\sum_{i=1}^n \|x_i\|_E^q\Big)^{1/q} \le K\Big\|\sum_{i=1}^n x_i\Big\|_E.$$

The quasi-Banach lattice E is said to be **order continuous** if for every downward directed set $(x_i)_{i\in I}$ in E with $\bigwedge_{i\in I} x_i = 0$, $\lim_i \|x_i\| = 0$. These notions will be used repeatedly in the sequel.

2.2. Non-commutative symmetric spaces. Assume that $\mathcal{M}$ is a semi-finite von Neumann algebra on a Hilbert space $\mathcal{H}$, equipped with a distinguished normal faithful semi-finite trace τ. If we set the definition ideal

$$\mathfrak{m}(\tau) := \Big\{\sum_{k=1}^n x_k y_k : n \in \mathbb{N}, \sum_{k=1}^n \tau(x_k^* x_k) < \infty, \sum_{k=1}^n \tau(y_k y_k^*) < \infty\Big\}$$

then $\mathfrak{m}(\tau)$ is dense in $\mathcal{M}$ for the weak operator topology and for $0 < p < \infty$, the non-commutative L^p-space associated with the pair $(\mathcal{M}, \tau)$ and denoted by $L^p(\mathcal{M}, \tau)$ is defined as the completion of $\mathfrak{m}(\tau)$ with respect to the (quasi-) norm

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p}$$

(here $|x| = (x^*x)^{1/2}$ is the usual modulus of x). When $1 \le p < \infty$, then $L^p(\mathcal{M}, \tau)$ is a Banach space, while for $0 < p < 1$, $L^p(\mathcal{M}, \tau)$ is only a p-Banach space.

It is now well-known that $L^p(\mathcal{M}, \tau)$'s can be realized as spaces of unbounded operators on $\mathcal{H}$. We recall the basic setup for further use. The identity element of $\mathcal{M}$ is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of all (self-adjoint) projections in $\mathcal{M}$. A closed densely defined operator a on $\mathcal{H}$ is said to be **affiliated with** $\mathcal{M}$ if $au = ua$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If a is a densely defined self-adjoint operator on $\mathcal{H}$, and if $a = \int_{-\infty}^{\infty} s\,de_s^a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{-\infty}^{\infty} \chi_B(s)\,de_s^a$. A closed densely defined operator a on $\mathcal{H}$ affiliated with $\mathcal{M}$ is said to be **τ-measurable** if there exists a number $s \ge 0$ such that $\tau(\chi_{(s,\infty)}(|a|)) < \infty$.

The set of all τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. The set $\widetilde{\mathcal{M}}$ is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [26]. For $\varepsilon, \delta > 0$, let

$$N(\varepsilon, \delta) = \Big\{x \in \widetilde{\mathcal{M}} : \text{for some } p \in \mathcal{P}(\mathcal{M}), \|xp\| < \varepsilon \text{ and } \tau(\mathbf{1} - p) \leq \delta\Big\}.$$

The system $\{N(\varepsilon, \delta); \varepsilon, \delta > 0\}$ forms a fundamental system of neighborhoods of the origin of the vector space $\widetilde{\mathcal{M}}$ and the translation-invariant topology induced by this system is called the **measure topology**. Equipped with the measure topology, $\widetilde{\mathcal{M}}$ is a complete topological *-algebra. These facts can be found in [26, 38]. Measure topology plays a very important role in this paper.

We recall the notion of generalized singular value function. For $x \in \widetilde{\mathcal{M}}$ and $t > 0$, we define

$$\mu_t(x) = \inf\big\{s \geq 0 : \tau(\chi_{(s,\infty)}(|x|)) \leq t\big\}, \quad \text{for } t \geq 0.$$

The function $t \mapsto \mu_t(x)$ from the interval $[0, \tau(\mathbf{1}))$ to $[0, \infty]$ is called the **generalized singular value function** of x. We refer the reader to [9] for an in depth study of $\mu(\cdot)$. We recall that if $\mathcal{M} = L^\infty(\mathbb{R}_+)$, then $\mu(f)$ is precisely the classical decreasing rearrangement of the function $|f|$. We also note that a sequence $(x_n)_{n\geq 1}$ in $\widetilde{\mathcal{M}}$ converges to 0 for the measure topology if and only if $\mu_t(x_n) \to_n 0$ for all $t > 0$.

To describe the general scheme of construction of general non-commutative spaces, we recall some basic definitions from general theory of rearrangement invariant spaces. We denote by $L^0(\mathbb{R}_+)$ the space of all $\mathbb{C}$-valued Lebesgue measurable functions defined on $\mathbb{R}_+$. A Banach space $(E, \|\cdot\|_E)$, where $E \subset L^0(\mathbb{R}_+)$, is called **rearrangement invariant Banach function space** if it follows from $f \in E$, $g \in L^0(\mathbb{R}_+)$ and $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. Furthermore, $(E, \|\cdot\|_E)$ is called **symmetric Banach function space** if it has the additional property that $f, g \in E$ and $g \prec\prec f$ imply that $\|g\|_E \leq \|f\|_E$. A symmetric Banach function space E is said to be **fully symmetric** if $f \in E, g \in L^0(\mathbb{R}_+)$ and $g \prec\prec f$ implies $g \in E$ and $\|g\|_E \leq \|f\|_E$. Here $g \prec\prec f$ denotes the submajorization in the sense of Hardy–Littlewood–Polya:

$$\int_0^t \mu_s(g)\ ds \leq \int_0^t \mu_s(f)\ ds, \quad \text{for all } t > 0.$$

We refer the reader to [2, 24] for any unexplained terminology from the general theory of rearrangement invariant function spaces and symmetric spaces.

Definition 2.1: Let $(E, \|\cdot\|_E)$ be a symmetric Banach function space on the interval $[0, \tau(\mathbf{1}))$. We define the **symmetric space of measurable operators**

$E(\mathcal{M},\tau)$ by setting:

$$E(\mathcal{M},\tau) = \{x \in \widetilde{\mathcal{M}} \ : \ \mu(x) \in E\} \quad \text{and}$$
$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E, \qquad \text{for } x \in E(\mathcal{M},\tau).$$

Equipped with $\|\cdot\|_{E(\mathcal{M},\tau)}$, the space $E(\mathcal{M},\tau)$ is a Banach space. It is often referred to as the noncommutative analogue of the function space E. Also, with obvious modifications, Definition 2.1 can be extended to include the case of some symmetric quasi-Banach function spaces (see [41] for details). More precisely, Definition 2.1 extends to quasi-Banach symmetric function spaces which are α-convex (with constant 1) for some $0 < \alpha \leq 1$. In this case $E(\mathcal{M},\tau)$ is only a α-Banach space and as in the case of function spaces, the inclusions

$$L^\alpha(\mathcal{M},\tau) \cap \mathcal{M} \subset E(\mathcal{M},\tau) \subset L^\alpha(\mathcal{M},\tau) + \mathcal{M}$$

hold with continuous embeddings (here $L^\alpha(\mathcal{M},\tau) \cap \mathcal{M}$ (respectively, $L^\alpha(\mathcal{M},\tau) + \mathcal{M}$) is equipped with the usual intersection (respectively, sum) (quasi-) norm of two (quasi-) Banach spaces). Extensive discussions on various properties of such spaces can be found in [5, 6, 29, 41]. In particular, if $E = L^p[0,\tau(\mathbf{1}))$, for $0 < p < \infty$, then $E(\mathcal{M},\tau)$ coincides with the noncommutative L^p-space associated with the pair $(\mathcal{M},\tau)$. Besides the L^p-spaces, Lorentz spaces play crucial role in our proofs. We collect below some definitions and basic facts about Lorentz spaces for further use.

For $0 < p < \infty$, $0 < q \leq \infty$, and $I = [0,1]$ or $\mathbb{R}_+$, the Lorentz function space $L_{p,q}(I)$ is the space of all $f \in L^0(I)$ for which $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} := \begin{cases} \left(\int_I \mu_t^q(f)\, d(t^{q/p}) \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t \in I} t^{1/p} \mu_t(f) & \text{if } q = \infty. \end{cases} \tag{2.1}$$

Clearly, $L_{p,p}(I) = L_p(I)$ for any $0 < p < \infty$. It is known that if $1 \leq q \leq p < \infty$, then (2.1) defines a norm under which $L_{p,q}(I)$ is a separable rearrangement invariant Banach function space. For the other cases, (2.1) defines only a quasinorm on $L_{p,q}(I)$ (see, for instance, [24]). If $1 < p \leq \infty$, the space $L^{p,\infty}(I)$ equipped with the equivalent Calderon norm $\|\cdot\|_{(p,\infty)}$ given by

$$\|f\|_{(p,\infty)} := \sup_{t \in I} \left\{ t^{1/p-1} \int_0^t \mu_s(f)\, ds \right\}, \quad f \in L^{p,\infty}(I),$$

is a symmetric Banach function space on I with the Fatou property. This norm will be used in the sequel. Noncommutative Lorentz spaces can be defined according to Definition 2.1.

For $1 < p < \infty$ and $1/p + 1/q = 1$, the following duality will be used in the sequel,

$$\big(L^{p,1}(\mathcal{M},\tau)\big)^* = L^{q,\infty}(\mathcal{M},\tau), \quad \text{(with equivalent norms).} \tag{2.2}$$

All interpolation results involving (classical) Lorentz spaces transfer verbatim to the noncommutative analogs (see, for instance, [29, Corollary 2.2, p. 1467]).

For $x \in \widetilde{\mathcal{M}}$, the right (respectively, left) support projections of x are denoted by $r(x)$ (respectively, $l(x)$).

Definition 2.2: (i) Two operators $x, y \in \widetilde{\mathcal{M}}$ are said to be **disjoint** if they have disjoint right and left supports: $r(x)r(y) = 0$ and $l(x)l(y) = 0$.

(ii) A sequence $(x_n)_{n\geq 1}$ in $\widetilde{\mathcal{M}}$ is called **disjoint** if the x_n's are pairwise disjoint.

(iii) A sequence $(x_n)_{n\geq 1}$ in $E(\mathcal{M},\tau)$ is called **almost disjoint** if there is a disjoint sequence $(x_n')_{n\geq 1}$ such that $\lim_{n\to\infty} \|x_n - x_n'\|_{E(\mathcal{M},\tau)} = 0$.

We remark that if $(x_n)_{n\geq 1}$ is an almost disjoint basic sequence in $E(\mathcal{M},\tau)$ then $(x_n)_{n\geq 1}$ is equivalent to a disjointly supported sequence in the function space E. A proof of this fact can be found in [30].

The next concept plays a very crucial role throughout the paper.

Definition 2.3: Let E be a symmetric quasi-Banach function space on the interval $[0, \tau(\mathbf{1}))$. We say that a subspace X of $E(\mathcal{M},\tau)$ is **strongly embedded** into $E(\mathcal{M},\tau)$ if the $\|\cdot\|_{E(\mathcal{M},\tau)}$-topology and the measure topology coincide on X.

PROPOSITION 2.4 ([30, Proposition 3.3]): *Let E be a symmetric quasi-Banach function space on $[0,\tau(\mathbf{1}))$ that is order continuous. Suppose that E is α-convex with constant 1 for some $0 < \alpha \leq 1$ and satisfies a lower q-estimate with constant 1 for some $q \geq 1$. If X is a subspace of $E(\mathcal{M},\tau)$ then one of the following statements holds:*

(i) *X is strongly embedded into $E(\mathcal{M},\tau)$;*
(ii) *X contains a normalized almost disjoint sequence.*

We should note that if $\mathcal{M}$ is finite then the two alternatives in Proposition 2.4 are exclusive but when $\mathcal{M}$ is infinite, it is possible to have disjointly supported basic sequence whose closed linear span is strongly embedded.

The next result shows that when E is order continuous a large class of subspaces of $E(\mathcal{M},\tau)$ can be embedded into $L^p(\mathcal{M},\tau)$ for appropriate values of p. This plays an important role in our proof in the next section.

Proposition 2.5: *Suppose that E is an order continuous and α-convex symmetric quasi-Banach function space on $\mathbb{R}_+$ and assume that E satisfies a lower q-estimate with constant* 1*. If Y is a separable subspace of $E(\mathcal{M},\tau)$ then either:*

(i) *Y contains an almost disjoint basic sequence; or*
(ii) *Y embeds isomorphically into $L^\alpha(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)$. In particular, if E is an order continuous symmetric Banach function space of $\mathbb{R}_+$ then Y embeds isomorphically into $\mathcal{M}_*\oplus_1\mathcal{M}_*$.*

Proof. Since Y is separable, we may assume without loss of generality that $\mathcal{M}$ is σ-finite. Indeed, choose a mutually orthogonal family $(f_i)_{i\in I}$ of projections in $\mathcal{M}$ with $\sum_{i\in I} f_i = \mathbf{1}$ for the strong operator topology and $\tau(f_i)<\infty$ for all $i\in I$. Let $\{y_n : n\geq 1\}$ be a countable dense subset of the unit ball of Y. Then using similar argument as in [41], one can get an at most countable subset $(f_k)_{k\in\mathbb{N}}$ of $(f_i)_{i\in I}$ such that for each f_i outside of $(f_k)_{k\in\mathbb{N}}$, $f_iy_n = y_nf_i = 0$ for every $n\geq 1$. Let $e=\sum_{k\in\mathbb{N}} f_k$ (for the strong operator topology). Then e is countably decomposable and therefore $e\mathcal{M}e$ is σ-finite. Since $ey_n = y_ne = y_n$ for every $n\geq 1$, it follows that $Y\subset L^p(e\mathcal{M}e)$. Replacing $\mathcal{M}$ by $e\mathcal{M}e$ and τ by its restriction on $e\mathcal{M}e$, we may assume that $e=\mathbf{1}$ and thus we can assume that $\mathcal{M}$ is σ-finite.

Suppose that Y does not contain any almost disjoint basic sequence. Then by Proposition 2.4, Y is strongly embedded into $E(\mathcal{M},\tau)$. Choose a mutually orthogonal countable family $(f_k)_{k\in\mathbb{N}}$ of finite projections in $\mathcal{M}$ with $\tau(f_k)<\infty$ for every $n\geq 1$ described above and for each $n\geq 1$, set

$$e_n := \sum_{k=1}^{n} f_k. \tag{2.3}$$

Then $e_n \uparrow_n \mathbf{1}$ and $\tau(e_n)<\infty$ for every $n\geq 1$. Define the following operator

$$\Theta_n : E(\mathcal{M},\tau)\to E(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)$$

as follows: if $x \in E(\mathcal{M},\tau)$, then

$$\Theta_n(x) = (e_n x, x e_n).$$

We note first that operation is well-defined. Indeed, for $x \in E(\mathcal{M},\tau)$, is it easy to verify that the operator $(e_n x, x e_n)$ is $\tau \oplus_\infty \tau$-measurable (as an unbounded operator on $\mathcal{H} \oplus_2 \mathcal{H}$). Moreover, from the definition of generalized singular value function, for $t > 0$ and a $\tau \oplus_\infty \tau$-measurable operator (a,b) then

$$\mu_t\big((a,b)\big) = \inf\big\{s \geq 0 : \tau(\chi_{(s,\infty)}(|a|)) + \tau(\chi_{(s,\infty)}(|b|)) \leq t\big\}$$

and, therefore,

$$\max\{\mu_t(a); \mu_t(b)\} \leq \mu_t\big((a,b)\big). \tag{2.4}$$

Moreover, since $\mu_t\big((a,0)\big) = \mu_t(a)$ and $\mu_t\big((0,b)\big) = \mu_t(b)$, it follows that

$$\mu\big((a,b)\big) \prec\prec \mu(a) + \mu(b). \tag{2.5}$$

Thus (2.4) and (2.5) imply that

$$\begin{aligned} \max\{\|a\|_{E(\mathcal{M},\tau)}; \|b\|_{E(\mathcal{M},\tau)}\} &\leq \|(a,b)\|_{E(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)} \\ &\leq \big(\|a\|^\alpha_{E(\mathcal{M},\tau)} + \|b\|^\alpha_{E(\mathcal{M},\tau)}\big)^{1/\alpha}. \end{aligned} \tag{2.6}$$

From these facts, it follows that for $n \geq 1$,

$$\|\Theta_n(x)\|_{E(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)} \leq 2\|x\|_{E(\mathcal{M},\tau)} \quad \forall x \in E(\mathcal{M},\tau),$$

which verifies that Θ_n is well-defined and bounded. The next step is to show that when restricted to Y, one of the Θ_n's must be an isomorphism.

Lemma 2.6: *There exists $n_0 \in \mathbb{N}$ so that the linear subspace $\Theta_{n_0}(Y)$ is a (strongly embedded) closed subspace of $E(\mathcal{M} \oplus_\infty \mathcal{M}, \tau \oplus_\infty \tau)$ and is isomorphic to Y.*

Assume by way of contradiction that for every $n \geq 1$, $\Theta_n|_Y$ is not an isomorphism. We will construct a sequence $(x_k)_{k\geq 1}$ in the unit sphere of Y and a strictly increasing sequence of integers $(n_k)_{k\geq1}$ satisfying

$$\big\|x_k - (e_{n_k} - e_{n_{k-1}})x_k(e_{n_k} - e_{n_{k-1}})\big\|^\alpha \leq 2^{-k}. \tag{2.7}$$

This would be in contradiction with the initial assumption that Y does not contain any almost disjoint basic sequence. The construction is done by induction. Fix $x_1 \in X$ with $\|x_1\| = 1$ and choose $n_1 \in \mathbb{N}$ so that $\big\|x_1 - e_{n_1}x_1e_{n_1}\big\| \leq 2^{-1}$. Suppose that $\{x_1, \dots, x_{k-1}\}$ and $n_1 < n_2 < \cdots < n_{k-1}$ were chosen that satisfy

(2.7). Since $\Theta_{n_{k-1}}|_Y$ is not an isomorphism, there exists $x_k \in Y$ with $\|x_k\| = 1$ and $\|\Theta_{n_{k-1}}(x_k)\|^\alpha \leq 2^{-(k+2)}$. A fortiori, (2.6) gives $\|e_{n_{k-1}} x_k\|^\alpha + \|x_k e_{n_{k_1}}\|^\alpha \leq 2^{-(k+1)}$. We can choose $n_k > n_{k-1}$ so that $\|x_k - e_{n_k} x_k e_{n_k}\|^\alpha \leq 2^{-(k+1)}$ (this is possible according to [7, Proposition 1.1] since E is order continuous). Combining the last two inequalities, we conclude

$$\begin{aligned}\|x_k - (e_{n_k} - e_{n_{k-1}}) x_k (e_{n_k} - e_{n_{k-1}})\|^\alpha &\\ \leq &\|x_k - e_{n_k} x_k e_{n_k}\|^\alpha + \|e_{n_k} x_k e_{n_{k-1}}\|^\alpha + \|e_{n_{k-1}} x_k (e_{n_k} - e_{n_{k-1}})\|^\alpha \\ \leq &\|x_k - e_{n_k} x_k e_{n_k}\|^\alpha + \|x_k e_{n_{k-1}}\|^\alpha + \|e_{n_{k-1}} x_k\|^\alpha \\ \leq &2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.\end{aligned}$$

The construction is complete. Hence, n_0 can be chosen so that $\Theta_{n_0}|_Y$ is an isomorphism. The fact that $\Theta_{n_0}(Y)$ is strongly embedded follows immediately from Proposition 2.4 since it does not contain any basic sequence equivalent to a disjointly supported sequence in E. This proves the lemma.

We conclude the proof by showing that $\Theta_{n_0}(Y)$ is a subspace $L^\alpha(\mathcal{M} \oplus_\infty \mathcal{M}, \tau \oplus_\infty \tau)$. This follows from Hölder's inequality. Indeed, since $\tau(e_{n_0}) < \infty$, $e_{n_0} \in L^\alpha(\mathcal{M}, \tau) \cap \mathcal{M}$. It follows that for every $x \in E(\mathcal{M}, \tau)$,

$$\begin{aligned}\|\Theta_{n_0}(x)\|_\alpha^\alpha &= \|e_{n_0} x\|_\alpha^\alpha + \|x e_{n_0}\|_\alpha^\alpha \\ &\leq \|e_{n_0}\|^\alpha_{L^\alpha(\mathcal{M},\tau)\cap\mathcal{M}} \big(\|e_{n_0} x\|^\alpha_{L^\alpha(\mathcal{M},\tau)+\mathcal{M}} + \|x e_{n_0}\|^\alpha_{L^\alpha(\mathcal{M},\tau)}\big) \\ &\leq \tau(e_{n_0}) \big(\|e_{n_0} x\|^\alpha_{E(\mathcal{M},\tau)} + \|x e_{n_0}\|^\alpha_{E(\mathcal{M},\tau)}\big) \\ &\leq 2\tau(e_{n_0}) \|\Theta_{n_0}(x)\|^\alpha_{E(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)} \\ &\leq 4\tau(e_{n_0}) \|x\|^\alpha_{E(\mathcal{M},\tau)}.\end{aligned}$$

This shows that $\Theta_{n_0}(Y) \subset L^\alpha(\mathcal{M} \oplus_\infty \mathcal{M}, \tau \oplus_\infty \tau)$. Since $\Theta_{n_0}(Y)$ is strongly embedded in $E(\mathcal{M} \oplus_\infty \mathcal{M}, \tau \oplus_\infty \tau)$, it is also a closed subspace of $L^\alpha(\mathcal{M} \oplus_\infty \mathcal{M}, \tau \oplus_\infty \tau)$. The proof is complete. ∎

Remark 2.7: In the preceding proposition the use of direct sum can be avoided if either of the following two cases is satisfied:

(i) $\mathcal{M}$ is finite and τ is a normal tracial state, or

(ii) $\mathcal{M}$ is a semi-finite properly infinite von Neumann algebra.

The first case is trivial since when $\mathcal{M}$ is finite, then it follows that whenever E is α-convex we have the inclusion $E(\mathcal{M}, \tau) \subset L^\alpha(\mathcal{M}, \tau)$. In this case, if Y is a subspace of $E(\mathcal{M}, \tau)$ that is strongly embedded then since the $\|\cdot\|_\alpha$-topology is weaker than $\|\cdot\|_{E(\mathcal{M},\tau)}$-topology and is stronger than the measure topology, strong embedding implies that the two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{E(\mathcal{M},\tau)}$ are equivalent when restricted to Y.

The second case follows from the general fact that if $\mathcal{M}$ is properly infinite then $\mathcal{M}\oplus_\infty\mathcal{M}$ is isomorphic to $\mathcal{M}$. Indeed, there exist two isometries u_1, u_2 in $\mathcal{M}$, such that $u_1u_1^*$ and $u_2u_2^*$ are orthogonal projections with $u_1u_1^*+u_2u_2^*=\mathbf{1}$. Define

$$\Phi:\mathcal{M}\to\mathcal{M}\oplus_\infty\mathcal{M}\quad\text{by}\quad\Phi(x)=(u_1^*x,u_2^*x)$$

and

$$\Psi:\mathcal{M}\oplus_\infty\mathcal{M}\to\mathcal{M}\quad\text{by}\quad\Psi(x,y)=u_1x+u_2x.$$

It is clear that $\Phi\circ\Psi=Id_{\mathcal{M}\oplus_\infty\mathcal{M}}$ and $\Psi\circ\Phi=Id_{\mathcal{M}}$. The same maps can be used to verify that if $0<p<\infty$ then $L^p(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)$ is isomorphic to $L^p(\mathcal{M},\tau)$.

An immediate application of Proposition 2.5 is the following known form of a noncommutative extension of the classical Kadec–Pełczynski dichotomy (see also [30, 32]).

Corollary 2.8: *Assume that E is an order continuous symmetric Banach function space on $\mathbb{R}_+$ and $E(\mathcal{M},\tau)$ is of Rademacher type 2. Then every subspace of $E(\mathcal{M},\tau)$ either contains an almost disjoint basic sequence or is isomorphic to a Hilbert space.*

Proof. Let X be a subspace of $E(\mathcal{M},\tau)$. If X does not contain any almost disjoint basic sequence then Proposition 2.5 implies that X embeds isomorphically into $L^1(\mathcal{M}\oplus_\infty\mathcal{M},\tau\oplus_\infty\tau)$. Therefore X is of Rademacher cotype 2. Since $E(\mathcal{M},\tau)$ is of Rademacher type 2, the conclusion that X is isomorphic to a Hilbert space follows from [23]. ∎

2.3. Haagerup L^p-spaces. There are several equivalent methods of constructing noncommutative L^p-spaces associated with general von Neumann algebra (see, e.g., [1, 11, 14, 15, 22, 31]). In this paper we will use Haagerup's construction ([11, 38]). Let us now provide a brief description of Haagerup's theory. Assume that $\mathcal{R}$ is a general von Neumann algebra (not necessarily semi-finite) on a Hilbert space H. For $0<p<\infty$, noncommutative L^p-spaces associated with $\mathcal{R}$ are defined as spaces of measurable operator relative to a larger semi-finite von Neumann algebra.

Fix a normal faithful semi-finite weight φ on $\mathcal{R}$ and let $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ be the one-parameter modular automorphism group associated with φ. We consider the crossed product $\mathcal{U} := \mathcal{R} \rtimes_{\sigma^\varphi} \mathbb{R}$, which is a von Neumann subalgebra of the algebra $\mathcal{B}(L^2(\mathbb{R}, H))$ generated by

$$\pi(x)(\xi(t)) = \sigma^\varphi_{-t}(x)(\xi(t)) \quad \text{and} \quad \lambda(s)(\xi(t)) = \xi(t-s)$$

for $t \in \mathbb{R}$ and $\xi \in L^2(\mathbb{R}, H)$. If $W(s)$ is the unitary operator on $L^1(\mathbb{R}, H)$ defined by

$$W(s)(\xi(t)) = e^{-ist}\xi(t),$$

then the dual action θ on $\mathcal{U}$ is given by

$$\theta_s(x) = W(s)xW(s)^*, \quad x \in \mathcal{U}.$$

The von Neumann algebra $\mathcal{R}$ can be identified with the subalgebra of $\mathcal{U}$

$$\pi(\mathcal{R}) = \big\{x \in \mathcal{U} : \theta_s(x) = x \text{ for all } s \in \mathbb{R}\big\}.$$

Moreover, it is known that $\mathcal{U}$ is semi-finite and is equipped with its canonical normal faithful semi-finite trace τ satisfying,

$$\tau \circ \theta_s = e^{-s}\tau, \quad s \in \mathbb{R}.$$

As above, the algebra of τ-measurable operators associated to the pair $(\mathcal{U}, \tau)$ is denoted by $\widetilde{\mathcal{U}}$. For $0 < p < \infty$, the Haagerup L^p-space associated with $\mathcal{R}$ is defined as a subset of the collection of τ-measurable operators by setting

$$L^p(\mathcal{R}, \varphi) := \big\{x \in \widetilde{\mathcal{U}} : \theta_s(x) = e^{-s/p}x,\, s \in \mathbb{R}\big\}.$$

This is clearly a closed self-adjoint linear subspace of $\widetilde{\mathcal{U}}$ and the norm is defined based on a known fact from [11, 38] that there is a linear homeomorphism $\psi \mapsto h_\psi$ from $\mathcal{R}_*$ onto $L^1(\mathcal{R}, \varphi)$. One can define a distinguished positive linear functional Tr on $L^1(\mathcal{R}, \varphi)$ by setting

$$\mathrm{Tr}(\mathrm{h}_\psi) = \psi(\mathbf{1}), \quad \psi \in \mathcal{R}_*.$$

Let $0 < p < \infty$ and $x \in L^p(\mathcal{R}, \varphi)$. If $x = u|x|$ is the polar decomposition of x, then $u \in \mathcal{R}$ and $|x| \in L^p(\mathcal{R}, \varphi)$. In particular, $|x|^p \in L^1(\mathcal{R}, \varphi)$. The (quasi-)norm on $L^p(\mathcal{R}, \varphi)$ is defined by:

$$\|x\|_p := \left(\mathrm{Tr}(|\mathrm{x}|^{\mathrm{p}})\right)^{1/p}, \quad \text{for } x \in L^p(\mathcal{R}, \varphi).$$

Equipped with $\|\cdot\|_p$, the space $L^p(\mathcal{R}, \varphi)$ is a Banach space (respectively, a p-Banach space) when $1 \le p < \infty$ (respectively, $0 < p < 1$). Some remarks are in order: (i) the space $L^p(\mathcal{R},\varphi)$ is independent of the particular choice of the normal faithful semi-finite weight φ on $\mathcal{R}$ (used in the construction of the crossed product) up to isometry so we will simply write $L^p(\mathcal{R})$ for $L^p(\mathcal{R},\varphi)$, (ii) the functional Tr on $\mathcal{R}$ and the canonical trace τ on $\mathcal{U}$ are quite different, (iii) when $\mathcal{R}$ is semi-finite then the Haagerup space $L^p(\mathcal{R})$ is isometric to the usual noncommutative L^p-space described previously. The reader is referred to [11, 38] for full details of Haagerup's theory.

It is shown in [9, Lemma 4.8] that if $x \in L^p(\mathcal{R})$, $0 < p < \infty$, then

$$\mu_t(x) = t^{-1/p}\|x\|_p, \quad t > 0, \tag{2.8}$$

where the generalized singular value function is taken relative to the pair $(\mathcal{U},\tau)$. This implies, in particular, that for any sequence in $L^p(\mathcal{R})$, the $\|\cdot\|_p$-convergence and the convergence in measure relative to $\widetilde{\mathcal{U}}$ coincide. This fact can also be found in [38, p. 40]. For later reference, we record the next proposition which is a direct consequence of (2.8) and the definition of weak-L^p-spaces. It will be used as a first step in reducing general L^p-spaces to those associated with semi-finite von Neumann algebras. This will be the only fact about Haagerup L^p-spaces that we will need for the rest of this paper.

Proposition 2.9: *If $0 < p < \infty$ then the Haagerup space $L^p(\mathcal{R})$ is a closed subspace of the space $L^{p,\infty}(\mathcal{U},\tau)$ associated with the semi-finite von Neumann algebra $(\mathcal{U},\tau)$. Moreover,*

$$\|x\|_p = \|x\|_{L^{p,\infty}(\mathcal{U},\tau)},$$

for all $x \in L^p(\mathcal{R})$.

3. Embeddings of Haagerup L^p-spaces when $0 < p < 2$

The principal result of this paper is the following theorem which can be viewed as extension of Junge's result to Haagerup L^p-spaces.

Theorem 3.1: *Let $\mathcal{R}$ be a σ-finite von Neumann algebra (not necessarily semi-finite). There exists a finite von Neumann algebra $\mathcal{M}$ equipped with a normal, tracial, faithful state τ such that for every $0 < r < p < 2$, there exists an isomorphic embedding of the Haagerup space $L^p(\mathcal{R})$ into $L^r(\mathcal{M},\tau)$. In particular for $1 < p < 2$, then $L^p(\mathcal{R})$ embeds as Banach space into $\mathcal{M}_*$.*

The following proposition is the decisive step toward our main result.

PROPOSITION 3.2: *Let $0 < q < p < 2$ and $\mathcal{W}$ be a σ-finite type-III von Neumann algebra. There exists a semi-finite von Neumann algebra $\mathcal{N}$ equipped with a normal faithful semi-finite trace σ so that whenever X is a subspace of $L^q(\mathcal{W})$ satisfying one of the following conditions:*

(i) *$1 \leq q < p < 2$ and X is a Banach space of Rademacher type p, or*
(ii) *$0 < q < p < 1$ and X is a p-Banach space.*

Then for every $0 < r < q$, X is isomorphic to a subspace of $L^r(\mathcal{N}, \sigma)$.

Proof. Let $\mathcal{U} = \mathcal{W} \rtimes_{\sigma^\varphi} \mathbb{R}$ be the crossed product of $\mathcal{W}$ by the modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ associated with a fixed faithful normal state φ. Denote by τ the canonical trace on the semi-finite von Neumann algebra $\mathcal{U}$ as described in the previous section. If X is a subspace of $L^q(\mathcal{W})$ then from Proposition 2.9, X is a closed subspace of $L^{q,\infty}(\mathcal{U}, \tau)$. Moreover, (2.8) implies that X is strongly embedded into $L^{q,\infty}(\mathcal{U}, \tau)$. Our aim is to prove that X embeds isomorphically into $L^r(\mathcal{U}, \tau)$ when $0 < r < q$. In particular, the semi-finite von Neumann algebra in the statement of the proposition can be taken to be $\mathcal{N} := \mathcal{U}$ and $\sigma := \tau$.

The argument consists of two steps. First, we embed X into a symmetric (quasi-) Banach space of measurable operators with order continuous norm and second, we show that with such embedding Proposition 2.5 above applies.

For the first step, fix $0 < p_1 < q < p_2 < p < 2$. We observe from [3, Theorem 5.3.1] that the Lorentz space $L^{q,\infty}(\mathbb{R}_+)$ is an interpolation space of the interpolation couple $(L^{p_1}(\mathbb{R}_+), L^{p_2}(\mathbb{R}_+))$. In particular, $L^{q,\infty}(\mathbb{R}_+) \subset L^{p_1}(\mathbb{R}_+) + L^{p_2}(\mathbb{R}_+)$. Here $L^{p_1}(\mathbb{R}_+) + L^{p_2}(\mathbb{R}_+)$ is the symmetric (quasi-) Banach function space on $\mathbb{R}_+$ equipped with the usual sum (quasi-) norm. Therefore $L^{q,\infty}(\mathcal{U}, \tau) \subset L^{p_1}(\mathcal{U}, \tau) + L^{p_2}(\mathcal{U}, \tau)$. Hence X can be viewed as a strongly embedded subspace of the symmetric space of measurable operators $L^{p_1}(\mathcal{U}, \tau) + L^{p_2}(\mathcal{U}, \tau)$.

We observe that the property of X (being either of Rademacher type strictly larger than q or being a p-Banach space) implies that it does not contain any basic sequence equivalent to the unit vector basis of l^q and therefore should not contain any almost disjoint sequence in the sense of $L^q(\mathcal{W})$ (i.e considering support projections from $\mathcal{W}$). The crucial property of X is that the same property is valid when it is viewed as a subspace of $L^{p_1}(\mathcal{U}, \tau) + L^{p_2}(\mathcal{U}, \tau)$. We state this in the next lemma.

Lemma 3.3: *The space X does not contain any basic sequence equivalent to a mutually disjoint sequence in $L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)$.*

The argument below for the verification of Lemma 3.3 is based on consideration of Rademacher types. Assume the contrary: there exists a normalized disjointly supported sequence $(y_n)_{n\geq 1}$ in $L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)$ that is equivalent to a sequence in X. Set $Y := \overline{span}\{y_n; n\geq 1\}$. Then by isomorphism either Y is a Banach space of Rademacher type p or Y is a p-Banach space according to $1<p<2$ or $0<p<1$.

For each $n\geq 1$, let $q_n := l(y_n)$ and $e_n := r(y_n)$ be the left and right support projection of y_n respectively. Then both $(q_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ are mutually disjoint sequences of projections and for every $n\geq 1$, $y_n = q_n y_n e_n$. For any finite sequence of scalars $(a_n)_{n\geq 1}$,

$$\begin{aligned}\Big|\sum_{n\geq 1} a_n y_n\Big|^2 &= \Big(\sum_{n\geq 1}\bar{a}_n e_n y_n^* q_n\Big)\Big(\sum_{n\geq 1} a_n q_n y_n e_n\Big)\\ &= \sum_{n\geq 1}|a_n|^2 e_n y_n^* q_n y_n e_n\\ &= \Big|\sum_{n\geq 1} a_n |y_n|\Big|^2.\end{aligned}$$

Therefore, $(y_n)_{n\geq 1}$ is equivalent to the basic sequence $(|y_n|)_{n\geq 1}$. Note that $(|y_n|)_{n\geq 1}$ is left and right disjointly supported by the sequence $(e_n)_{n\geq 1}$. For each $n\geq 1$, the semi-finiteness of e_n guaranties the existence of an increasing family $\{e_\beta^{(n)}\}_\beta$ of projections in the von Neumann algebra $e_n\mathcal{U}e_n$ with $\tau(e_\beta^{(n)})<\infty$ for every β and satisfies $0\leq e_\beta^{(n)} \uparrow^\beta e_n$. Since $L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)$ is order-continuous, it follows that

$$\lim_\beta \big\|e_\beta^{(n)}|y_n|e_\beta^{(n)} - |y_n|\big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} = 0.$$

Thus for any given $\varepsilon>0$, a projection $\widetilde{e}_n\leq e_n$ can be chosen such that $\tau(\widetilde{e}_n)<\infty$ and

$$\big\|\widetilde{e}_n|y_n|\widetilde{e}_n - |y_n|\big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} \leq \varepsilon. \tag{3.1}$$

We observe that $(\widetilde{e}_n)_{n\geq 1}$ is a sequence of mutually disjoint of finite projections and if $(a_n)_{n\geq 1}$ is a finite sequence of scalars, then we have

$$\begin{aligned}\Big\|\sum_{n\geq 1} a_n\widetilde{e}_n|y_n|\widetilde{e}_n\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} &\leq \Big\|\sum_{n\geq 1} a_n|y_n|\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} \\ &= \Big\|\sum_{n\geq 1} a_n y_n\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)}.\end{aligned} \tag{3.2}$$

The first inequality follows from the fact that if $\widetilde{e} := \sum_{n\geq 1}\widetilde{e}_n$ for the strong operator topology, then $\sum_{n\geq 1} a_n\widetilde{e}_n|y_n|\widetilde{e}_n = \widetilde{e}\big(\sum_{n\geq 1} a_n|y_n|\big)\widetilde{e}$.

If $\alpha_1 = 0$ and $\alpha_n = \sum_{i=1}^n \tau(\tilde{e}_i) < \infty$, set $f_n := \mu_{(\cdot)-\alpha_{n-1}}(\widetilde{e}_n|y_n|\widetilde{e}_n)$ for $n \geq 1$. The sequence of positive functions $(f_n)_{n\geq 1}$ is disjointly supported in $L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)$ and (as basic sequence) is equivalent to the sequence $(\widetilde{e}_n|y_n|\widetilde{e}_n)_{n\geq 1}$. Therefore (3.2) implies

$$\Big\|\sum_{n\geq 1} a_n f_n\Big\|_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)} \leq \Big\|\sum_{n\geq 1} a_n y_n\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)}. \tag{3.3}$$

Fix $\delta > 0$ and consider $f^{(1)}$ and $f^{(2)}$ in $L^{p_1}(\mathbb{R}_+)$ and $L^{p_2}(\mathbb{R}_+)$ respectively with:

(1) $\sum_{n\geq 1} a_n f_n = f^{(1)} + f^{(2)}$;
(2) $\|f^{(1)}\|_{p_1} + \|f^{(2)}\|_{p_2} \leq \|\sum_{n\geq 1} a_n f_n\|_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)} + \delta$.

Denote by $(A_n)_{n\geq 1}$ the mutually disjoint sequence of measurable subsets of $\mathbb{R}_+$ consisting of supports of $(f_n)_{n\geq 1}$. Then for $n \geq 1$, $a_n f_n = \chi_{A_n} f^{(1)} + \chi_{A_n} f^{(2)}$ and item (2) above combined with (3.3) imply the inequality,

$$\begin{aligned}\Big(\sum_{n\geq 1}\|\chi_{A_n} f^{(1)}\|_{p_1}^{p_1}\Big)^{1/p_1} &+ \Big(\sum_{n\geq 1}\|\chi_{A_n} f^{(2)}\|_{p_2}^{p_2}\Big)^{1/p_2} \\ &\leq \Big\|\sum_{n\geq 1} a_n y_n\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} + \delta.\end{aligned}$$

Since $p_1 \leq p_2$, we get

$$\Big(\sum_{n\geq 1}(\|\chi_{A_n} f^{(1)}\|_{p_1} + \|\chi_{A_n} f^{(2)}\|_{p_2})^{p_2}\Big)^{1/p_2} \leq \Big\|\sum_{n\geq 1} a_n y_n\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} + \delta.$$

A fortiori,

$$\Big(\sum_{n\geq 1}\|a_n f_n\|^{p_2}_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)}\Big)^{1/p_2} \leq \Big\|\sum_{n\geq 1} a_n y_n\Big\|_{L^{p_1}(\mathcal{U},\tau)+L^{p_2}(\mathcal{U},\tau)} + \delta.$$

Since δ is arbitrary, we obtain that

$$\Big(\sum_{n\geq 1} \|a_n f_n\|_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)}^{p_2}\Big)^{1/p_2} \leq \Big\|\sum_{n\geq 1} a_n y_n\Big\|_Y.$$

Recall that since $(y_n)_{n\geq 1}$ is disjointly supported, it is an unconditional basic sequence and hence,

$$\Big(\sum_{n\geq 1} \|a_n f_n\|_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)}^{p_2}\Big)^{1/p_2} \leq \Big(\int_0^1 \Big\|\sum_{n\geq 1} a_n r_n(t) y_n\Big\|_Y^{1/2}\Big)^{1/2}.$$

Since Y is either a Banach space of Rademacher type p or is p-convex, there exists a constant $T(Y,p)$ such that

$$\Big(\sum_{n\geq 1} \|a_n f_n\|_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)}^{p_2}\Big)^{1/p_2} \leq T(Y,p)\Big(\sum_{n\geq 1} \|a_n y_n\|_Y^p\Big)^{1/p}. \tag{3.4}$$

The fact that $(y_n)_{n\geq 1}$ is a normalized sequence and $\|f_n\|_{L^{p_1}(\mathbb{R}_+)+L^{p_2}(\mathbb{R}_+)} \geq 1-\varepsilon$ (see (3.1) above) with (3.4) implies that for any finite sequence $(a_n)_{n\geq 1}$ of scalars,

$$(1-\varepsilon)\|(a_n)\|_{l^{p_2}} \leq T(Y,p)\|(a_n)\|_{l^p}.$$

Taking limit as $\varepsilon \to 0$, we conclude that

$$\|(a_n)\|_{l^{p_2}} \leq T(Y,p)\|(a_n)\|_{l^p}.$$

This is a contradiction since $p_2 < p < 2$. The lemma is verified.

To conclude the proof of Proposition 3.2, we consider two cases.

- Assume that $0 < q \leq 1$. Then $0 < p_1 < 1$ and the symmetric quasi-Banach function space $L^{p_1}(\mathbb{R}_+) + L^{p_2}(\mathbb{R}_+)$ is p_1-convex. Now since X does not contain any almost disjoint basic sequence from $L^{p_1}(\mathcal{U},\tau) + L^{p_2}(\mathcal{U},\tau)$, Proposition 2.5 applied to X and the symmetric (quasi-) Banach function space $E = L^{p_1}(\mathbb{R}_+) + L^{p_2}(\mathbb{R}_+)$. Therefore, X embeds isomorphically into $L^{p_1}(\mathcal{U} \oplus_\infty \mathcal{U}, \tau \oplus_\infty \tau)$. Moreover, the von Neumann algebra $\mathcal{U}$ is semi-finite and properly infinite (see, for instance, [21, p. 985]). It follows from Remark 2.7 that X embeds isomorphically into $L^{p_1}(\mathcal{U},\tau)$. Taking $p_1 = r$, the proof is complete for the case $0 < q \leq 1$.
- For the case $1 < p < 2$, we can take $p_1 = 1$ and deduce that X embeds isomorphically into $L^1(\mathcal{U},\tau)$. Moreover, X is reflexive. We can invoke Theorem A.2 from the appendix below to conclude that X embeds isomorphically into $L^r(\mathcal{U},\tau)$.

Combining the two cases, the proof of Proposition 3.2 is complete. ∎

Proof of Theorem 3.1. We may assume without loss of generality that $\mathcal{R}$ is a type-III von Neumann algebra. The proof combines Junge's result stated in Theorem 1.1 together with its type-III counterpart (Corollary 1.2) and Proposition 3.2.

Fix $0 < r < s < s+\varepsilon < p < 2$. By Corollary 1.2, there exists a von Neumann algebra $\mathcal{W}$ so that $L^p(\mathcal{R})$ embeds isometrically into $L^{s+\varepsilon}(\mathcal{W})$. Let X be a subspace of $L^{s+\varepsilon}(\mathcal{W})$ isometric to $L^p(\mathcal{R})$. If $1 \leq p < 2$ then X is a Banach space of Rademacher type $p > s+\varepsilon$ (see [8]) and if $0 < p < 1$ then X is a p-Banach space. Moreover, as $\mathcal{R}$ is σ-finite, $L^p(\mathcal{R})$ is separable and so is X. Hence, Proposition 3.2 applies to X and $L^{s+\varepsilon}(\mathcal{W})$. There exists a semi-finite von Neumann algebra $\mathcal{N}$ equipped with a semi-finite trace σ so that X embeds isomorphically into $L^s(\mathcal{N},\sigma)$. The conclusion follows from another application of Theorem 1.1 to the semi-finite von Neumann algebra $\mathcal{N}$. There exists a finite von Neumann algebra $\mathcal{M}$ equipped with a faithful normal tracial state τ so that $L^s(\mathcal{N},\sigma)$ embeds isometrically into $L^r(\mathcal{M},\tau)$. Thus combining all three isomorphisms above, we can conclude that $L^p(\mathcal{R})$ embeds isomorphically into $L^r(\mathcal{M},\tau)$. The proof is complete. ∎

We remark that the embedding in Theorem 3.1 only deals with isomorphism. We do not know if one can obtain a similar result for isometric embedding. The following question is still open.

Problem: Let M and N be von Neumann algebras. If M is finite and $L^p(N)$ embeds isometrically into $L^q(M)$ for all $0 < q < p < 2$, is N semi-finite?

As pointed out in the introduction, for $1 \leq q < p \leq 2$, Theorem 3.1 cannot be improved to the case of operator spaces. That is, if the von Neumann algebra $\mathcal{R}$ is type-III then the (Banach) isomorphism in Theorem 3.1 can not be taken to be a complete embedding. This can be seen from results of Xu ([40]): denote by C_p (respectively, R_p) the subspace of the Schatten class S_p consisting of matrices whose entries are zero except those in the first column (respectively, row). From [40, Theorem 5.4, Theorem 5.6], if $1 \leq q < p \leq 2$, there exists a type-III factor $\mathcal{R}$ such that C_p and R_p embed completely isomorphically into $L^q(\mathcal{R})$ but neither C_p nor R_p embeds completely isomorphically into any noncommutative L^q-spaces associated with semi-finite von Neumann algebras. In particular, $L^q(\mathcal{R})$

does not completely embed into any L^r-space associated with any semi-finite von Neumann algebra for $1 \leq r < q$.

As application of Theorem 3.1, we obtained in connection with a recent result of Junge and Parcet [19] the following structural property of reflexive subspaces of preduals of general von Neumann algebras:

THEOREM 3.4: *Let $\mathcal{R}$ be a von Neumann algebra. There exists a finite von Neumann algebra $\mathcal{M}$ so that every reflexive subspace of $\mathcal{R}_*$ Banach embeds isomorphically into $\mathcal{M}_*$.*

Proof. Let X be a reflexive subspace of $\mathcal{R}_*$. According to [19], there exists $1 < p < 2$ so that X embeds isomorphically into $L^p(\mathcal{R} \oplus_\infty \mathcal{R})$. Apply Theorem 3.1 to the von Neumann algebra $\mathcal{R} \oplus_\infty \mathcal{R}$ in order to get a finite von Neumann algebra $\mathcal{M}$ so that $L^p(\mathcal{R} \oplus_\infty \mathcal{R})$ embeds isomorphically into $\mathcal{M}_*$. ∎

Appendix A. Reflexive subspaces of preduals of semi-finite von Neumann algebras

In this section, we provide a noncommutative analog of Rosenthal's classical theorem [33]. A well-known result of Rosenthal on Banach structures of classical L^p-spaces reads as follows:

THEOREM A.1 ([33, Theorem 8]): *If $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space, $1 \leq p < 2$, and R is a closed linear subspace of $L^p(\mu)$. Then either R contains a complemented copy of l^p or there exists $p' > p$ such that R embeds isomorphically into $L^{p'}(\mu)$. In particular, every reflexive subspace of $L^1(\mu)$ embeds isomorphically into $L^r(\mu)$ for some $r > 1$.*

More precisely, if $1 < r < 2$ and X is a subspace of $L^1(\mu)$ which is of Rademacher type strictly larger than r, then X embeds isomorphically into $L^r(\mu)$.

Our aim is to provide suitable generalizations of Theorem A.1 for the setting of noncommutative L^p-spaces associated with semi-finite von Neumann algebra. Recall that such generalizations have been considered in the literature. The first of such noncommutative generalizations is a result of Friedman [10] asserting that for $1 \leq p < \infty$, any subspace of the Schatten class S^p either contains a copy of l^p or is isomorphic to a Hilbert space. A general result in the spirit of Rosenthal's theorem goes back to Pisier [28]. For the remaining of this section, we assume that $\mathcal{M}$ is semi-finite von Neumann algebra equipped with a distinguished normal faithful semi-finite trace τ. The next theorem is the principal result of this section.

THEOREM A.2: *Let $1 < r < 2$ and X be a subspace of $L^1(\mathcal{M},\tau)$ of Rademacher type strictly larger than r. Then X embeds isomorphically into $L^r(\mathcal{M},\tau)$. In particular, if R is a reflexive subspace of $L^1(\mathcal{M},\tau)$ then there exists $1 < r < 2$ so that R embeds isomorphically into $L^r(\mathcal{M},\tau)$.*

Very recently, Theorem A.2 has been generalized by Junge and Parcet to include the general Haagerup L^p-spaces (see [19]). The proof for the semi-finite case given below, however, is much simpler in comparison with the general case and therefore deserves a separate consideration.

Our approach follows essentially the argument of Pisier in [28] and is based on the following factorization of operators on C^*-algebras. We only record here the parts that we need.

THEOREM A.3 ([28, Theorem 3.2]): *Let $2 \leq q < \infty$ and Y be a Banach space of Rademacher cotype q. If $T : \mathcal{M} \to Y$ is a bounded linear operator then there exist an absolute constant $C > 0$ and a positive functional $\varphi \in \mathcal{M}^*$ such that for every $a \in \mathcal{M}$,*

$$\|Ta\| \leq C\varphi(aa^* + a^*a)^{1/q}\|a\|_\infty^{1-(2/q)}.$$

Moreover, if Y is a dual Banach space and T is weak-continuous then φ can be taken from $\mathcal{M}_*$.*

We start with the following structural lemma for subspaces of (semi-finite) non commutative L^1-spaces.

MAIN LEMMA: *Let $1 < r < 2$. If X is a subspace of $L^1(\mathcal{M},\tau)$ of Rademacher type r then there exists a positive operator $b \in \mathcal{M}$ with the following properties:*

(i) *$\|b\|_\infty \leq 1$ and b^{-1} exists in $L^1(\mathcal{M},\tau) + \mathcal{M}$;*
(ii) *the set $bXb := \{bxb; x \in X\}$ is a closed linear subspace of $L^{r,\infty}(\mathcal{M},\tau)$ and is isomorphic to X.*

Proof. Assume $1 < r < 2$ and $2 < r' < \infty$ such that $1/r + 1/r' = 1$. Since X is of Rademacher type r, it does not contain any copy of l^1 and therefore is strongly embedded into $L^1(\mathcal{M},\tau)$ (see Proposition 2.4). Moreover, its dual X^* is of Rademacher cotype r' (see, e.g., [24, Proposition 1.e.17, p. 79]). Let

$J : X \to L^1(\mathcal{M}, \tau)$ be the formal inclusion map (i.e. $J(x) = x$ for all $x \in X$). The adjoint map $J^* : \mathcal{M} \to X^*$ satisfies the assumption of Theorem A.3. Therefore there exist a constant $C > 0$ and $\varphi \in \mathcal{M}_*$ such that for every $a \in \mathcal{M}$,

$$\|J^*(a)\| \leq C\varphi(aa^* + a^*a)^{1/r'} \|a\|_\infty^{1-(2/r')}. \tag{A.1}$$

As $L^1(\mathcal{M}, \tau)$ is identified with the predual of $\mathcal{M}$, there exists $v \in L^1(\mathcal{M}, \tau)_+$ such that $\varphi(a) = \tau(va)$ for every $a \in \mathcal{M}$. Therefore, (A.1) becomes: for every $a \in \mathcal{M}$,

$$\|J^*(a)\| \leq C\tau\big(v(aa^* + a^*a)\big)^{1/r'} \|a\|_\infty^{1-(2/r')}. \tag{A.2}$$

Consider the operator $v + \mathbf{1} \in L^1(\mathcal{M}, \tau) + \mathcal{M}$. Since $v \geq 0$, the operator $v + \mathbf{1}$ is invertible and $(v + \mathbf{1})^{-1} \in \mathcal{M}$ with $\|(v + \mathbf{1})^{-1}\|_\infty \leq 1$. Define

$$b := (v + \mathbf{1})^{-1/2}. \tag{A.3}$$

We claim that the operator $b \in \mathcal{M}$ as defined above satisfies the conclusion of the main lemma. For this, consider the map $M : L^1(\mathcal{M}, \tau) \cap \mathcal{M} \to \mathcal{M}$ as the right and left multiplications by b, that is,

$$M(a) = bab \quad \forall a \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}.$$

The next lemma shows that the normal functional φ in (A.1) can be replaced by the trace τ when we use the map J^*M in place of J^*.

Lemma A.4: *There exists a constant $C' > 0$ such that*

$$\|J^*M(a)\| \leq C'\|a\|_2^{2/r'} \|a\|_\infty^{1-(2/r')},$$

for every $a \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$.

It suffices to verify this lemma for self-adjoint element of $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$. Assume that $a = a^*$. Since b is also self-adjoint, $|bab|^2 = bab^2ab \leq ba^2b$. Then from (A.2), we get

$$\|J^*M(a)\| \leq C\tau\big(v(ba^2b)\big)^{1/r'} \|bab\|_\infty^{1-(2/r')} \leq C\tau\big((bvb)a^2\big)^{1/r'} \|a\|_\infty^{1-(2/r')}.$$

Since $bvb \leq \mathbf{1}$ by construction, the inequality in the statement is verified for self-adjoint elements. The general statement easily follows with $C' = 2C$. Thus Lemma A.4 is verified.

We now proceed with the proof of the main lemma. It follows from Lemma A.4 and general interpolation technique (see, e.g., [3, p. 49]) that there is a constant

$C" > 0$ such that for every $a \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$,

$$\|J^*M(a)\| \leq C" \|a\|_{r',1}. \tag{A.4}$$

Since $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ is dense in $L^{r',1}(\mathcal{M},\tau)$, from (A.4), there exists an operator $L : L^{r',1}(\mathcal{M},\tau) \to X^*$ which generates the following commutative diagram:

$$\begin{array}{ccc} L^1(\mathcal{M},\tau) \cap \mathcal{M} & \xrightarrow{M} & \mathcal{M} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle J^*} \\ L^{r',1}(\mathcal{M},\tau) & \xrightarrow{L} & X^* \end{array}$$

where i is the inclusion map. Taking the adjoint maps, the duality (2.2) gives

$$\begin{array}{ccc} X & \xrightarrow{J} & L^1(\mathcal{M},\tau) \\ {\scriptstyle L^*}\downarrow & & \downarrow{\scriptstyle M^*} \\ L^{r,\infty}(\mathcal{M},\tau) & \xrightarrow{i^*} & L^1(\mathcal{M},\tau) + \mathcal{M}. \end{array}$$

We observe first that $M^*(y) = byb$ for every $y \in L^1(\mathcal{M},\tau)$. Moreover, M^*J is an isomorphism. Indeed, the map $\Delta : L^1(\mathcal{M}) + \mathcal{M} \to \widetilde{\mathcal{M}}$ defined by $\Delta(z) = b^{-1}zb^{-1}$ is continuous and $\Delta M^*J(x) = J(x) = x$ for all $x \in X$. Since X strongly embeds into $L^1(\mathcal{M},\tau)$, the map ΔM^*J is an isomorphism and so is the map $M^*J = i^*L^*$. A fortiori, L^* is an isomorphism of X into $L^{r,\infty}(\mathcal{M},\tau)$. To conclude the proof, it remains to verify that $L^*(X) = bXb$. To see this, we observe that i^* is again the inclusion map so for every $x \in X$, $L^*(x) = i^*L^*(x) = M^*J(x) = bxb$ as measurable operator. The proof of the main lemma is complete. ∎

Proof of Theorem A.2. Let X be a subspace of $L^1(\mathcal{M},\tau)$ of Rademacher type $s > r$. ¿From the main lemma, there exists $b \in \mathcal{M}$ with $\|b\|_\infty \leq 1$ and bXb is a closed subspace of $L^{s,\infty}(\mathcal{M},\tau)$. We observe that bXb is also isomorphic to X as a subspace of $L^{s,\infty}(\mathcal{M},\tau) \cap L^1(\mathcal{M},\tau)$. Indeed, the map $\Lambda : L^{s,\infty}(\mathcal{M},\tau) \cap L^1(\mathcal{M},\tau) \to \widetilde{\mathcal{M}}$ defined by $y \mapsto \Lambda(y) = b^{-1}yb^{-1}$ is one to one and takes bXb onto X. Since X is strongly embedded, $\Lambda\mid_{bXb}$ is an isomorphism. We conclude that since $L^{s,\infty}(\mathcal{M},\tau) \cap L^1(\mathcal{M},\tau) \subset L^r(\mathcal{M},\tau)$(see, e.g., [24, p. 143]), strong embedding again implies that bXb is a subspace of $L^r(\mathcal{M},\tau)$ isomorphic to X.

Assume now that R is a reflexive subspace of $L^1(\mathcal{M},\tau)$ then from [16], R is superreflexive and therefore it has non-trivial Rademacher type. ∎

Extensions to the case $1 < p < 2$ goes as follows:

Corollary A.5: *Let $1 < p < 2$ then every subspace Y of $L^p(\mathcal{M}, \tau)$ either contains a complemented subspace isomorphic to l^p, or Y embeds into $L^r(\mathcal{M}, \tau)$ for some $r > p$.*

Proof. Assume that $1 < p < 2$ and let Y be a subspace of $L^p(\mathcal{M}, \tau)$. Then from Proposition 2.4, either Y contains a complemented subspace isomorphic to l^p or Y is strongly embedded into $L^p(\mathcal{M}, \tau)$. Let

$$p_Y := \sup\{r : Y \text{ is of Rademacher type } r\}.$$

We claim that *if Y contains no (complemented) copy of l^p (and therefore is strongly embedded into $L^p(\mathcal{M}, \tau)$) then $p_Y > p$.* Indeed, it is known from [25] that l^{p_Y} is finitely represented in Y. Therefore, if $p_Y = p$, then Y contains l_n^p's uniformly and by [13, Theorem 4.4], Y contains a copy of l^p which is a contradiction. Hence $p_Y > p$. Let $\varepsilon > 0$ and $r > 0$ so that $p < r < r + \varepsilon < p_Y$. Then Y is of Rademacher type $r+\varepsilon$ and we claim that Y embeds into $L^r(\mathcal{M}, \tau)$.

The von Neumann algebra $\mathcal{M}$ admits a decomposition

$$\mathcal{M} = \mathcal{M}_1 \oplus_\infty \mathcal{M}_2$$

where $\mathcal{M}_1$ is a finite von Neumann algebra and $\mathcal{M}_2$ is a semi-finite properly infinite von Neumann algebra (see for instance [34, Theorem 2.2.3]). Let τ_1 be a faithful normal tracial state on $\mathcal{M}_1$. Then

$$\begin{aligned} L^p(\mathcal{M}, \tau) &\approx L^p(\mathcal{M}_1, \tau|_{\mathcal{M}_1}) \oplus_p L^p(\mathcal{M}_2, \tau|_{\mathcal{M}_2}) \\ &\approx L^p(\mathcal{M}_1, \tau_1) \oplus_p L^p(\mathcal{M}_2, \tau|_{\mathcal{M}_2}) \\ &\approx L^p(\mathcal{M}_1 \oplus_\infty \mathcal{M}_2, \tau_1 \oplus_\infty \tau|_{\mathcal{M}_2}). \end{aligned}$$

Therefore, the space Y can be viewed as a (strongly embedded) subspace of the noncommutative space $L^p(\mathcal{M}_1 \oplus_\infty \mathcal{M}_2, \tau_1 \oplus_\infty \tau|_{\mathcal{M}_2})$. Fix an increasing sequence of finite projections $(e_n)_{n\geq 1}$ in $\mathcal{M}_2$ with $\tau(e_n) < \infty$ for every $n \geq 1$ and $e_n \uparrow^n \mathbf{1}_{\mathcal{M}_2}$. Define for every $n \geq 1$,

$$\Theta_n : L^p(\mathcal{M}_1 \oplus_\infty \mathcal{M}_2, \tau_1 \oplus_\infty \tau|_{\mathcal{M}_2}) \to L^p(\mathcal{M}_1 \oplus_\infty \mathcal{M}_2 \oplus_\infty \mathcal{M}_2, \tau_1 \oplus_\infty \tau|_{\mathcal{M}_2} \oplus_\infty \tau|_{\mathcal{M}_2})$$

by

$$\Theta_n(x, y) := (x, e_n y, y e_n).$$

Similar argument as the one used in the proof of Proposition 2.5 above shows that there exists $n_0 \in \mathbb{N}$ so that $\Theta_{n_0}|_Y$ is an isomorphism and $\Theta_{n_0}(Y)$ is strongly embeded into $L^p(\mathcal{M}_1 \oplus_\infty \mathcal{M}_2 \oplus_\infty \mathcal{M}_2, \tau_1 \oplus_\infty \tau|_{\mathcal{M}_2} \oplus_\infty \tau|_{\mathcal{M}_2})$ (details of this are left to the interested reader). Since τ_1 is finite and $\tau(e_{n_0}) < \infty$, it follows that $\Theta_{n_0}(Y) \subset L^1(\mathcal{M}_1 \oplus_\infty \mathcal{M}_2 \oplus_\infty \mathcal{M}_2, \tau_1 \oplus_\infty \tau|_{\mathcal{M}_2} \oplus_\infty \tau|_{\mathcal{M}_2})$. Moreover, since $\mathcal{M}_2$ is properly infinite, $L^1(\mathcal{M}_2 \oplus_\infty \mathcal{M}_2, \tau|_{\mathcal{M}_2} \oplus_\infty \tau|_{\mathcal{M}_2}) \approx L^1(\mathcal{M}_2, \tau|_{\mathcal{M}_2})$, we conclude that Y is isomorphic to a subspace of $L^1(\mathcal{M}, \tau)$. The conclusion follows immediately from Theorem A.2. ∎

An immediate application of Proposition 3.2 extends Corollary A.5 to the case of general von Neumann algebras.

Corollary A.6: *Let $\mathcal{R}$ be a σ-finite von Neumann algebra and $(\mathcal{N}, \nu)$ be the pair of semi-finite von Neumann algebra with the normal, faithful semi-finite trace ν from Proposition 3.2. If $1 < p < 2$, then every subspace Y of $L^p(\mathcal{R})$ either contains a complemented subspace isomorphic to l^p, or embeds isomorphically into $L^r(\mathcal{N}, \nu)$ for some $r > p$.*

Proof. The proof follows the argument used in Corollary A.5. If Y does not contain any complemented copy of l^p then from [32, Theorem 5.1], we can deduce as in the proof of Corollary A.5 that Y is of Rademacher type strictly larger than p. Since Y embeds isomorphically into $L^1(\mathcal{N}, \nu)$, the conclusion follows from Theorem A.2. ∎

We conclude with a remark on the case of Schatten-classes. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators in the Hilbert space $\mathcal{H}$, then the following result (due to Friedmann [10] for $1 \le p < \infty$) holds: *for $0 < p < \infty$, every subspace of the Schatten-class $S^p(\mathcal{H})$ either contain a further subspace equivalent to l^p or is isomorphic to a Hilbert space.* For the range $0 < p \le 2$, this follows immediately from Proposition 2.4 above. Indeed, it is known that in this case $\widetilde{\mathcal{M}} = \mathcal{B}(\mathcal{H})$ and the measure topology coincides with norm operator topology in $\mathcal{B}(\mathcal{H})$. Moreover, if $0 < p \le 2$, $S^p(\mathcal{H}) \subset S^2(\mathcal{H})$ and therefore every strongly embedded subspace of $S^p(\mathcal{H})$ is a closed subspace of $S^2(\mathcal{H})$. The range $2 < p < \infty$ can be viewed as a particular case of the noncommutative generalization of the classical Kadec-Pełczynski theorem (see Corollary 2.8 above).

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